Lecture 4: Review of Linear Algebra

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Outline

Vectors

- A vector is an ordered set of numbers arranged in a column. An n-dimensional column vector **a** is

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Matrices

- A matrix is a set of column vectors. An $n \times k$ matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix}$$

Square and symmetric matrix

A square matrix the number of rows equal the number of columns, that is, $n = k$

A symmetric matrix the (i, j) element equal to the (j, i) element.

Diagonal matrix

- A diagonal matrix: a square matrix in which all off-diagonal elements equal zero, that is,

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Identity matrix

- An identity matrix: a diagonal matrix in which all diagonal elements are 1. A subscript is sometimes included to indicate its size, e.g. $\mathbf{I}_4$ indicate a 4×4 identity matrix.

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Triangular matrix

- A triangular matrix: have only zeros either above or below the main diagonal. A lower triangular matrix looks like

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Transpose

- The transpose of a matrix $\mathbf{A}$, denoted $\mathbf{A}'$, is obtained by creating the matrix whose kth row is the kth column of the original matrix. That is,

$$\mathbf{B} = \mathbf{A}' \Leftrightarrow b_{ik} = a_{ki} \text{ for all } i \text{ and } k$$

- For any $\mathbf{A}$, we have $(\mathbf{A}')' = \mathbf{A}$
- If $\mathbf{A}$ is symmetric, then $\mathbf{A} = \mathbf{A}'$.

Addition

- For two matrices **A** and **B** with the same dimensions, that is both are $n \times k$.

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}] \text{ for all } i \text{ and } j$$

Vector multiplication

- The inner product of two $n \times 1$ column vector $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}'\mathbf{b} = \sum_{i=1}^{n} a_i b_i$$

Since both $\mathbf{a}$ and $\mathbf{b}$ are $n \times 1$ vectors, it must hold that $\mathbf{a}'\mathbf{b} = \mathbf{b}'\mathbf{a}$.

Matrix multiplication

- Suppose that $\mathbf{A}$ is an $n \times m$ matrix and $\mathbf{B}$ is an $m \times k$ matrix, then the product $\mathbf{C} = \mathbf{AB}$ is an $n \times k$ matrix, where the (i, j) element of $\mathbf{C}$ is $c_{ij} = \sum_{l=1}^{m} a_{il} b_{lj}$.
  In other words, if we write $\mathbf{A}$ and $\mathbf{B}$ with vectors, that is,

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \\ \vdots \\ \mathbf{a}_n' \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix}$$

  where $\mathbf{a}_i = [a_{i1}, a_{i2}, \cdots, a_{im}]'$ is the i$^{\text{th}}$ row of $\mathbf{A}$ for $i = 1, 2, \ldots, n$, and $\mathbf{b}_j = [b_{1j}, b_{2j}, \ldots, b_{mj}]'$ is the j$^{\text{th}}$ column of $\mathbf{B}$ for $j = 1, 2, \ldots, k$.

Matrix multiplication (cont'd)

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1'\mathbf{b}_1 & \cdots & \mathbf{a}_1'\mathbf{b}_k \\ \mathbf{a}_2'\mathbf{b}_1 & \cdots & \mathbf{a}_2'\mathbf{b}_k \\ \vdots & \ddots & \vdots \\ \mathbf{a}_n'\mathbf{b}_1 & \cdots & \mathbf{a}_n'\mathbf{b}_k \end{bmatrix}$$

Properties of matrix addition and multiplication

- Commutative law: $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$. No commutative law for matrix multiplication.
- Associative law: $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$ and $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$
- Distributive law: $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$
- Transpose of a sum and a product: $(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$ and $(\mathbf{AB})'=\mathbf{B}'\mathbf{A}'$.

Definition

- Let $\mathbf{A}$ be an $n \times n$ square matrix. $\mathbf{A}$ is said to be invertible or nonsingular if such a matrix $\mathbf{A}^{-1}$ exists that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$. $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$.

Calculation

- Let a^{ik} be the ik$^{\text{th}}$ element of $\mathbf{A}^{-1}$. The general formula for computing an inverse matrix is

$$a^{ik} = \frac{|\mathbf{C}_{ki}|}{|\mathbf{A}|}$$

where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$, $|\mathbf{C}_{ki}|$ is the ki$^{\text{th}}$ cofactor of $\mathbf{A}$, that is, the determinant of the matrix $\mathbf{A}_{ki}$ obtained from $\mathbf{A}$ by deleting row k and column i, pre-multiplied by $(-1)^{(k+i)}$.

Example 1: The inverser of a 2×2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example 2: The inverse of a diagonal matrix

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}^{-1} = \begin{bmatrix} 1/a_{11} & 0 & \cdots & 0 \\ 0 & 1/a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}$$

Linear independence

- The set of k $n \times 1$ vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are linearly independent if there do not exist nonzero scalars $c_1, c_2, \ldots, c_k$ such that $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_k\mathbf{a}_k = \mathbf{0}_{n\times 1}$.

The rank of a matrix

- The rank of the $n \times k$ matrix $\mathbf{A}$ is the number of linearly independent column vectors of $\mathbf{A}$, denoted as $\mathrm{rank}(\mathbf{A})$.
- If $\mathrm{rank}(\mathbf{A}) = k$, then $\mathbf{A}$ is said to have full column rank. Then, there do not exist a nonzero $k \times 1$ vector $\mathbf{c}$ such that $\mathbf{Ac} = \mathbf{0}$.
- If $\mathbf{A}$ is an $n \times n$ square matrix and $\mathrm{rank}(\mathbf{A}) = n$, then $\mathbf{A}$ is nonsingular.
- If $\mathbf{A}$ has full column rank, then $\mathbf{A}'\mathbf{A}$ is nonsingular.

Positive definite matrices

- Let $\mathbf{V}$ be an $n \times n$ square matrix. Then $\mathbf{V}$ is positive definite if $\mathbf{c}'\mathbf{V}\mathbf{c} > 0$ for all nonzero $n \times 1$ vector $\mathbf{c}$.
- $\mathbf{V}$ is positive semidefinite if $\mathbf{c}'\mathbf{V}\mathbf{c} \geq 0$ for all nonzero $n \times 1$ vector $\mathbf{c}$.
- If $\mathbf{V}$ is positive definite, then it is nonsingular.

Calculus with Vectors and Matrices

- We need to use the following results of matrix calculus in the future lectures.

$$\frac{\partial \mathbf{a}'\mathbf{x}}{\partial \mathbf{x}} = \mathbf{a}, \ \frac{\partial \mathbf{x}'\mathbf{a}}{\partial \mathbf{x}} = \mathbf{a}, \quad \text{and}$$
$$\frac{\partial \mathbf{x}'\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}')\mathbf{x}$$

When $\mathbf{A}$ is symmetric, then $(\partial \mathbf{x}'\mathbf{A}\mathbf{x})/(\partial \mathbf{x}) = 2\mathbf{A}\mathbf{x}$