Lecture 4: Review of Linear Algebra

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Contents

In this part, we will go over some very basic knowledge of linear algebra, which will be used in this course. The main references are Appendix 18.1 in Stock and Watson's book.

1 Vectors and Matrices

1.1 Vectors

A **vector** is an ordered set of numbers arranged either in a row or in a column. An n-dimensional column vector **a** and an n-dimensional row vector **b** are

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1, b_2, \ldots, b_n \end{bmatrix}$$

1.2 Matrices

A **matrix** is a set of column vectors or a set of row vectors. An $n \times k$ matrix **A** is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix}$$

1.3 Types of Matrices

- A **square** matrix: the number of rows equal the number of columns, that is, $n = k$
- A **symmetric** matrix: the (i, j) element equal to the (j, i) element.
- A **diagonal** matrix: a square matrix in which all off-diagonal elements equal zero, that is,

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

- An **identity** matrix: a diagonal matrix in which all diagonal elements are 1. A subscript is sometimes included to indicate its size, e.g. $\mathbf{I}_4$ indicate a 4×4 identity matrix.

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- A **triangular** matrix: have only zeros either above or below the main diagonal. A **lower**

triangular matrix looks like

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

2 Matrix Operations

2.1 Transpose

The **transpose** of a matrix $\mathbf{A}$, denoted $\mathbf{A}'$, is obtained by creating the matrix whose kth row is the kth column of the original matrix. That is,

$$\mathbf{B} = \mathbf{A}' \Leftrightarrow b_{ik} = a_{ki} \text{ for all } i \text{ and } k$$

- For any $\mathbf{A}$, we have $(\mathbf{A}')' = \mathbf{A}$
- If $\mathbf{A}$ is symmetric, then $\mathbf{A} = \mathbf{A}'$.

2.2 Addition

For two matrices $\mathbf{A}$ and $\mathbf{B}$ with the same dimensions, that is both are $n \times k$.

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}] \text{ for all } i \text{ and } j$$

2.3 Multiplication

- **Vector multiplication**. The **inner product** of two $n \times 1$ column vector $\mathbf{a}$ and $\mathbf{b}$ is

  $$\mathbf{a}'\mathbf{b} = \sum_{i=1}^{n} a_i b_i$$

  Since both $\mathbf{a}$ and $\mathbf{b}$ are $n \times 1$ vectors, it must hold that $\mathbf{a}'\mathbf{b} = \mathbf{b}'\mathbf{a}$.

- **Matrix multiplication**. The matrices $\mathbf{A}$ and $\mathbf{B}$ can be multiplied if they are conformable, that is, if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$.

  Suppose that $\mathbf{A}$ is an $n \times m$ matrix and $\mathbf{B}$ is an $m \times k$ matrix, then the product $\mathbf{C} = \mathbf{AB}$ is an $n \times k$ matrix, where the (i, j) element of $\mathbf{C}$ is $c_{ij} = \sum_{l=1}^{m} a_{il} b_{lj}$.

In other words, if we write $\mathbf{A}$ and $\mathbf{B}$ with vectors, that is,

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \\ \vdots \\ \mathbf{a}_n' \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix}$$

where $\mathbf{a}_i = [a_{i1}, a_{i2}, \cdots, a_{im}]'$ is the i$^{\text{th}}$ row of $\mathbf{A}$ for $i = 1, 2, \ldots, n$, and $\mathbf{b}_j = [b_{1j}, b_{2j}, \ldots, b_{mj}]'$ is the j$^{\text{th}}$ column of $\mathbf{B}$ for $j = 1, 2, \ldots, k$. Then,

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1'\mathbf{b}_1 & \cdots & \mathbf{a}_1'\mathbf{b}_k \\ \mathbf{a}_2'\mathbf{b}_1 & \cdots & \mathbf{a}_2'\mathbf{b}_k \\ \vdots & \ddots & \vdots \\ \mathbf{a}_n'\mathbf{b}_1 & \cdots & \mathbf{a}_n'\mathbf{b}_k \end{bmatrix}$$

2.4 Properties of matrix addition and multiplication

- **Commutative law**: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. No commutative law for matrix multiplication.
- **Associative law**: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ and $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- **Distributive law**: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- **Transpose of a sum and a product**: $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$ and $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

3 Matrix Inverse

3.1 Definition

Let $\mathbf{A}$ be an $n \times n$ square matrix. $\mathbf{A}$ is said to be **invertible** or **nonsingular** if such a matrix $\mathbf{A}^{-1}$ exists that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$. $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$.

3.2 Calculation

Let a^{ik} be the ik$^{\text{th}}$ element of $\mathbf{A}^{-1}$. The general formula for computing an inverse matrix is

$$a^{ik} = \frac{|\mathbf{C}_{ki}|}{|\mathbf{A}|}$$

where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$, $|\mathbf{C}_{ki}|$ is the ki$^{\text{th}}$ cofactor of $\mathbf{A}$, that is, the determinant of the matrix $\mathbf{A}_{ki}$ obtained from $\mathbf{A}$ by deleting row k and column i, pre-multiplied by $(-1)^{(k+i)}$.

- Example 1. Calculate the inverse of a 2×2 matrix.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- Example 2. The inverse of a diagonal matrix.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}^{-1} = \begin{bmatrix} 1/a_{11} & 0 & \cdots & 0 \\ 0 & 1/a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}$$

4 Linear Independence

4.1 Linear independence

The set of k $n \times 1$ vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are **linearly independent** if there do not exist nonzero scalars $c_1, c_2, \ldots, c_k$ such that $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_k\mathbf{a}_k = \mathbf{0}_{n\times 1}$.

4.2 The rank of a matrix

The **rank** of the $n \times k$ matrix $\mathbf{A}$ is the number of linearly independent column vectors of $\mathbf{A}$, denoted as $\text{rank}(\mathbf{A})$.

- If $\text{rank}(\mathbf{A}) = k$, then $\mathbf{A}$ is said to have full column rank. Then, there do not exist a nonzero $k \times 1$ vector $\mathbf{c}$ such that $\mathbf{Ac} = \mathbf{0}$.
- If $\mathbf{A}$ is an $n \times n$ square matrix and $\text{rank}(\mathbf{A}) = n$, then $\mathbf{A}$ is nonsingular.
- If $\mathbf{A}$ has full column rank, then $\mathbf{A}'\mathbf{A}$ is nonsingular.

5 Positive Definite and Eigenvalues

5.1 Positive definite matrices

Let $\mathbf{V}$ be an $n \times n$ square matrix. Then $\mathbf{V}$ is **positive definite** if $\mathbf{c}'\mathbf{V}\mathbf{c} > 0$ for all nonzero $n \times 1$ vector $\mathbf{c}$. And $\mathbf{V}$ is positive semidefinite if $\mathbf{c}'\mathbf{V}\mathbf{c} \geq 0$ for all nonzero $n \times 1$ vector $\mathbf{c}$.

- If $\mathbf{V}$ is positive definite, then it is nonsingular.

5.2 Eigenvalues and eigenvectors

Let $\mathbf{A}$ be an $n \times n$ square matrix. If the $n \times 1$ vector $\mathbf{q}$ and the scalar λ satisfy $\mathbf{Aq} = \lambda\mathbf{q}$, where $\mathbf{q}'\mathbf{q} = 1$, then λ is the **eigenvalue** of $\mathbf{A}$ and $\mathbf{q}$ is the **eigenvector** of $\mathbf{A}$ associated with λ.

- If $\mathbf{V}$ is an $n \times n$ symmetric positive definite matrix, then all the eigenvalues of $\mathbf{V}$ are positive real numbers.

6 Calculus with Vectors and Matrix

We need to use the following results of matrix calculus in the future lectures.

$$\frac{\partial \mathbf{a}'\mathbf{x}}{\partial \mathbf{x}} = \mathbf{a}, \ \frac{\partial \mathbf{x}'\mathbf{a}}{\partial \mathbf{x}} = \mathbf{a}, \quad \text{and}$$
$$\frac{\partial \mathbf{x}'\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}')\mathbf{x}$$

when $\mathbf{A}$ is symmetric, then $(\partial \mathbf{x}'\mathbf{A}\mathbf{x})/(\partial \mathbf{x}) = 2\mathbf{A}\mathbf{x}$